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A Uniform Asymptotic Expansion for Krawtchouk Polynomials

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We study the asymptotic behavior of the Krawtchouk polynomial $K_n^{(N)}(x; p, q)$ as $n \to \infty$. With $x \equiv \lambda N$ and $v = n/N$, an infinite asymptotic expansion is derived, which holds uniformly for λ and v in compact subintervals of $(0, 1)$. This expansion involves the parabolic cylinder function and its derivative. When v is a fixed number, our result includes the various asymptotic approximations recently given by M. E. H. Ismail and P. Simeonov. © 2000 Academic Press

Key Words: Krawtchouk polynomials; uniform asymptotic expansion; parabolic cylinder function.

1. INTRODUCTION

Let $p > 0$, $q > 0$, and $p + q = 1$, and let N be a positive integer. By the binomial expansion, we have

$$(1 + qw)^x (1 - pw)^{N-x} = \sum_{n=0}^{\infty} K_n^{(N)}(x; p, q) w^n, \tag{1.1}$$

where

$$K_n^{(N)}(x; p, q) = \sum_{k=0}^{n} \binom{N-x}{n-k} \binom{x}{k} (-p)^{n-k} q^k. \tag{1.2}$$

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(Note that if $n > N$, then $K_n^{(N)}(x; p, q) = 0$ when $x = 0, 1, ..., N$.) Clearly, $K_n^{(N)}(x; p, q)$ is a polynomial in x of degree n. For convenience, we shall sometimes use the simpler notation

$$K_n(x) \equiv K_n^{(N)}(x; p, q). \tag{1.3}$$

These polynomials are known as the Krawtchouk polynomials, and they form an orthogonal system on the discrete set $\{0, 1, 2, ..., N\}$ with the weight function

$$\rho(x) = \binom{N}{x} p^x q^{N-x}, \qquad x = 0, 1, ..., N. \tag{1.4}$$

More precisely, we have

$$\sum_{j=0}^{N} K_n(j)\, K_m(j) \binom{N}{j} p^j q^{N-j} = \binom{N}{n} p^n q^n\, \delta_{n,m}, \qquad n, m = 0, 1, ..., N. \tag{1.5}$$

For a proof of (1.5), we refer to Szegö [13, p. 36]. For additional properties of these polynomials, see the references cited in [4, p. 161].

Recently, there is a considerable interest in the asymptotics of Krawtchouk polynomials, when the degree n grows to infinity. For instance, in [12], Sharapudinov has obtained the asymptotic formula

$$\begin{aligned}(2Npq\pi n!)^{1/2}\, (Npq)^{-n/2}\, \rho(\hat{x})\, e^{x^2/2} K_n(\hat{x}) \\ = e^{-x^2/2}(2^n n!)^{-1/2}\, H_n(x) + O(n^{7/4} N^{-1/2}),\end{aligned} \tag{1.6}$$

where $\hat{x} = Np + (2Npq)^{1/2}\, x$, $n = O(N^{1/3})$, $x = O(n^{1/2})$, and $H_n(x)$ is the Hermite polynomial. Furthermore, if the zeros of $K_n(\hat{x})$ are arranged in decreasing order, $\hat{x}_{1,N} > \hat{x}_{2,N} > \cdots > \hat{x}_{n,N}$, then he has shown that

$$\hat{x}_{n,N} = Np[1 - (2q/Np)^{1/2}\, x_1(n)] + O(n^{7/4})$$

uniformly with respect to $1 \leqslant n \leqslant \eta_N N^{1/4}$, $N = 1, 2, ...,$ where $\{\eta_N\}$ is a sequence of positive numbers tending to zero as $N \to \infty$ and $x_1(n)$ is the smallest zero of the Hermite polynomial. Properties of the zeros of Krawtchouk polynomials are important in the study of the Hamming scheme of classical coding theory; see [7, 9] and the references given there. Also, Ismail and Simeonov [5] have investigated the asymptotic behavior of $K_n(x)$ as $n \to \infty$, when $N/n = \gamma$ is a fixed constant independent of n. Their approach is based on the classical method of saddle point; see [11, pp. 125–127; 14, pp. 103–105]. In particular, when $p = q = \frac{1}{2}$, they have given an asymptotic formula for $K_n(nt)$ in each of the t-intervals: (a) $(0, \frac{1}{2}\gamma - \sqrt{\gamma - 1})$ when $1 < \gamma < 2$, (b) $(0, \frac{1}{2}\gamma - \sqrt{\gamma - 1})$ when $\gamma \geqslant 2$, and

(c) $(\frac{1}{2}\gamma - \sqrt{\gamma - 1}, \frac{1}{2}\gamma)$. Corresponding results for $\frac{1}{2}\gamma < t < \gamma$ can be obtained by using the symmetry formula

$$K_n^{(N)}(x; p, q) = (-1)^n K_n^{(N)}(N - x; q, p) \tag{1.7}$$

which follows readily from (1.2). Similar results have also been provided in [5] for the case $p \neq q$.

The purpose of this paper is to present a uniform asymptotic expansion for $K_n(x)$ in the interval $0 < x < N$, as $n \to \infty$. To make it more precise, we let $v = n/N$ and $x = \lambda N$. This choice makes $v \in (0, 1)$ and $\lambda \in (0, 1)$. We shall derive an infinite asymptotic expansion for $K_n(\lambda N)$ as $n \to \infty$, which holds uniformly for v and λ in any compact subinterval of $(0, 1)$. In a subsequent paper, this result will be used to construct asymptotic approximations for the zeros of Krawtchouk polynomials in various cases depending on the values of p, q, and v.

2. SADDLE POINTS

By Cauchy's integral formula, we have from (1.1) and (1.3)

$$K_n(x) = \frac{1}{2\pi i} \int_C (1 - pw)^{N-x} (1 + qw)^x \frac{dw}{w^{n+1}}, \tag{2.1}$$

where C is a small closed contour surrounding $w = 0$. For convenience, we put

$$\sigma := p/q, \tag{2.2}$$

and make pw in (2.1) as the integration variable so that

$$K_n(x) = \frac{p^n \sigma^{-x}}{2\pi i} \int_{C'} (1 - w)^{N-x} (\sigma + w)^x \frac{dw}{w^{n+1}}, \tag{2.3}$$

where we assume, without loss of generality, C' is a circle centered at the origin with a sufficiently small radius. As in Section 1, we set

$$\lambda := x/N \qquad \text{and} \qquad v := n/N. \tag{2.4}$$

The integral in (2.3) can then be written in the form

$$K_n(\lambda N) = \frac{p^n \sigma^{-\lambda N}}{2\pi i} \int_{C'} e^{NF(w, \lambda)} \frac{dw}{w}, \tag{2.5}$$

where the phase function $F(w, \lambda)$ is given by

$$F(w, \lambda) \equiv (1-\lambda)\log(1-w) + \lambda\log(\sigma+w) - v\log w. \tag{2.6}$$

For definiteness, we choose for all logarithmic functions in (2.6) the principal branch

$$\log\zeta = \log|\zeta| + i\arg\zeta, \qquad -\pi \leqslant \arg\zeta \leqslant \pi. \tag{2.7}$$

The function $F(w, \lambda)$ has branch points at $w=0$, $w=1$, $w=-\sigma$, and $w=\infty$, although $w=0$ is not a branch point of the integrand in (2.3) or, equivalently, (2.5). This function is analytic and single-valued in the complex w-plane with cuts along the intervals $(-\infty, 0]$ and $[1, \infty)$, or along the intervals $(-\infty, -\sigma]$ and $[0, \infty)$. For later discussion we also need to specify the values of the argument along the upper and lower edges of the cuts. To this end, we let $w^{\pm} = u + i0^{\pm}$, $u > 1$, denote points on the upper and lower edges of the cut along $[1, \infty)$. We shall choose

$$\arg(1-w^{+}) = -\pi \qquad \text{and} \qquad \arg(1-w^{-}) = \pi. \tag{2.8}$$

If $w^{\pm} = u + i0^{\pm}$, $u < -\sigma$, are points on the upper and lower edges of the cut along $(-\infty, -\sigma]$, then we choose

$$\arg(\sigma+w^{+}) = \pi \qquad \text{and} \qquad \arg(\sigma+w^{-}) = -\pi. \tag{2.9}$$

From (2.4), we know that the values of the parameters λ and v lie in the interval (0, 1). In view of the symmetry relation in (1.7), we may restrict ourselves to the case

$$0 < p \leqslant q < 1. \tag{2.10}$$

Thus, it follows from (2.2) that the value of the parameter σ lies in (0, 1]. That is, we have

$$0 < \lambda < 1, \qquad 0 < v < 1, \qquad 0 < \sigma \leqslant 1. \tag{2.11}$$

The saddle points of the phase function $F(w, \lambda)$ are easily found to be

$$w_{\pm}(\lambda) = \frac{[\lambda(1+\sigma) - \sigma - v + \sigma v] \pm \sqrt{[\lambda(1+\sigma) - \sigma - v + \sigma v]^2 - 4\sigma v(1-v)}}{2(1-v)}. \tag{2.12}$$

These points coalesce when λ and v satisfy

$$[\lambda(1+\sigma) - \sigma - v + \sigma v]^2 = 4\sigma v(1-v) \tag{2.13}$$

or, equivalently, when λ takes the values

$$\lambda_{\pm} \equiv \lambda_{\pm}(v) = \frac{(\sigma + v - \sigma v) \pm 2\sqrt{\sigma v(1-v)}}{1+\sigma} = \frac{(\sqrt{v} \pm \sqrt{\sigma(1-v)})^2}{1+\sigma}. \tag{2.14}$$

For simplicity, we have suppressed the dependence on v in (2.6) and (2.12). Since

$$(\sigma + v - \sigma v) + 2\sqrt{\sigma v(1-v)} \leqslant (\sigma + v - \sigma v) + \sigma v + (1-v) = 1 + \sigma,$$

we have

$$0 \leqslant \lambda_- < \lambda_+ \leqslant 1. \tag{2.15}$$

Note that

$$\lambda_- = 0 \qquad \text{if and only if} \qquad v = \frac{\sigma}{1+\sigma} = p \tag{2.16}$$

and

$$\lambda_+ = 1 \qquad \text{if and only if} \qquad v = \frac{1}{1+\sigma} = q. \tag{2.17}$$

Straightforward substitution of (2.14) in (2.12) gives

$$w_+(\lambda_{\pm}) = w_-(\lambda_{\pm}) = \pm\sqrt{\frac{\sigma v}{1-v}} \equiv \pm r_0. \tag{2.18}$$

We also note that since

$$\frac{\sigma v}{1-v} \lesseqqgtr 1 \qquad \text{if and only if} \qquad v \lesseqqgtr \frac{1}{1+\sigma} \tag{2.19}$$

and

$$\frac{\sigma v}{1-v} \lesseqqgtr \sigma^2 \qquad \text{if and only if} \qquad v \lesseqqgtr \frac{\sigma}{1+\sigma}, \tag{2.20}$$

we have

$$r_0 \gtreqless 1 \qquad \text{when} \qquad v \gtreqless q \tag{2.21}$$

and

$$r_0 \gtreqless \sigma \qquad \text{when} \qquad v \gtreqless p. \tag{2.22}$$

By examining the derivatives $w'_+(\lambda)$ and $w'_-(\lambda)$, we can determine the directions in which the saddle points $w_+(\lambda)$ and $w_-(\lambda)$ move, as λ varies from 0 to 1. The movements of $w_+(\lambda)$ and $w_-(\lambda)$ are shown in Figs. 1a–1d, and the "arrows" in these figures are used to indicate the directions of the movements. To be more specific, we make the following observations: (i) when $v > q$, $w_+(1) = \sigma v/(1-v)$ and $w_-(1) = 1$; (ii) when $v < q$, $w_+(1) = 1$ and $w_-(1) = \sigma v/(1-v)$; (iii) when $v = q$, $\lambda_+ = 1$ and $w_\pm(\lambda_+) = w_\pm(1) = 1$; (iv) when $v > p$, $w_+(0) = -\sigma$ and $w_-(0) = -v/(1-v)$; (v) when $v < p$, $w_+(0) = -v/(1-v)$ and $w_-(0) = -\sigma$; (vi) when $v = p$, $\lambda_- = 0$ and $w_\pm(\lambda_-) = w_\pm(0) = -\sigma$; (vii) when $\lambda_- < \lambda < \lambda_+$, $w_- = \bar{w}_+$ and $|w_\pm(\lambda)| = r_0$.

From Figs. 1a–1d, one readily sees that the movements of w_+ and w_- are very similar to those studied in [6] for the Meixner polynomials and in [8] for the Meixner–Pollaczek polynomials.

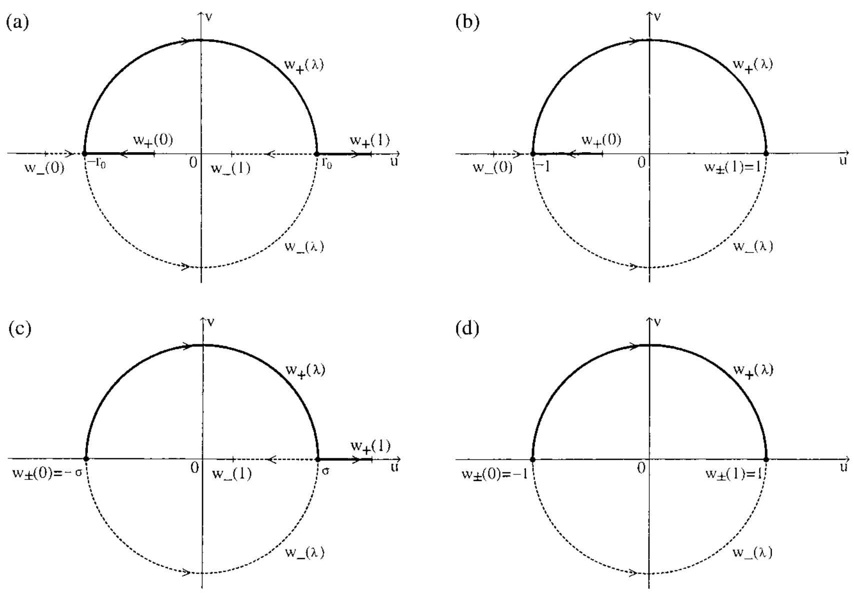


FIG. 1. (a) Movement of w_+ and w_- $(v \neq p, q)$. (b) Movement of w_+ and w_- $(v = q, v \neq p)$. (c) Movement of w_+ and w_- $(v = p, v \neq q)$. (d) Movement of w_+ and w_- $(v = p = q = \frac{1}{2})$.

3. STEEPEST DESCENT PATHS

To find the relevant steepest descent paths for the integral (2.5), we first consider the real part of the phase function in (2.6). Clearly, we have

$$\operatorname{Re} F(w, \lambda) = (1-\lambda) \log |1-w| + \lambda \log |\sigma + w| - \nu \log |w|. \tag{3.1}$$

Note that $\operatorname{Re} F \to +\infty$, as $w \to 0$ or as $w \to \infty$, and that $\operatorname{Re} F \to -\infty$, as $w \to 1$ or as $w \to -\sigma$. Hence, the relevant steepest descent paths must end at $w = 1$ or $w = -\sigma$, and not at $w = 0$ and $w = \infty$. Next we examine the function

$$\operatorname{Im} F(w, \lambda) = (1-\lambda) \arg(1-w) + \lambda \arg(\sigma + w) - \nu \arg w. \tag{3.2}$$

Our discussion will be divided into several cases.

Case (a). $\lambda_+ < \lambda < 1$ and $\nu < q$. In this case, $0 < w_- < w_+ < 1$. Since $\operatorname{Im} F(w_\pm) = 0$, the steepest descent paths through $w_\pm$ are given by

$$(1-\lambda) \arg(1-w) + \lambda \arg(\sigma + w) - \nu \arg w = 0.$$

Obviously, points in the interval $(0, 1)$ on the real axis satisfy this equation; see Fig. 2a.

Case (b). $\lambda_+ < \lambda < 1$ and $\nu > q$. Here, we have $1 < w_- < w_+$. Since there is a cut along the infinite half-line $(1, \infty)$, we let w^+ and w^- denote any point w on the upper and lower edge of the cut, respectively. From (2.8), it follows that

$$\operatorname{Im} F(w_\pm) = \begin{cases} (1-\lambda) \arg(1-w_\pm^+) = -\pi(1-\lambda) \\ (1-\lambda) \arg(1-w_\pm^-) = \pi(1-\lambda). \end{cases}$$

If we write $w = u + iv$, then the steepest descent paths through $w_\pm$ are given by

$$(1-\lambda) \arg(1-w) + \lambda \arg(\sigma + w) - \nu \arg w = \begin{cases} -\pi(1-\lambda) & \text{if} \quad v > 0 \\ \pi(1-\lambda) & \text{if} \quad v < 0. \end{cases}$$

Points on the straight line $(1, \infty)$ clearly satisfy this equation; see Fig. 2b.

Case (c). $\lambda_- < \lambda < \lambda_+$. Note that $w_- = \bar{w}_+$ and $F(w_-) = \overline{F(w_+)}$. Hence, $\operatorname{Im} F(w_\pm) = \pm \operatorname{Im} F(w_+)$ and the equation $\operatorname{Im} F(w) = \operatorname{Im} F(w_\pm)$

becomes $\operatorname{Im}[F(w) \mp F(w_+)] = 0$. The steepest descent paths are therefore given by

$$(1-\lambda)[\arg(1-w) \mp \arg(1-w_+)] + \lambda[\arg(\sigma+w) \mp \arg(\sigma+w_+)] - v[\arg w \mp \arg w_+] = 0;$$

see Fig. 2c.

Case (d). $0 < \lambda < \lambda_-$ and $v < p$. Here we have $-\sigma < w_- < w_+ < 0$ and $\operatorname{Im} F(w_\pm) = -v \arg w_\pm = -v\pi$. Thus the steepest descent paths are given by

$$(1-\lambda)\arg(1-w) + \lambda \arg(\sigma+w) - v \arg w = -v\pi,$$

and the points on the line segment $(-\sigma, 0)$ clearly satisfy this equation; see Fig. 2d.

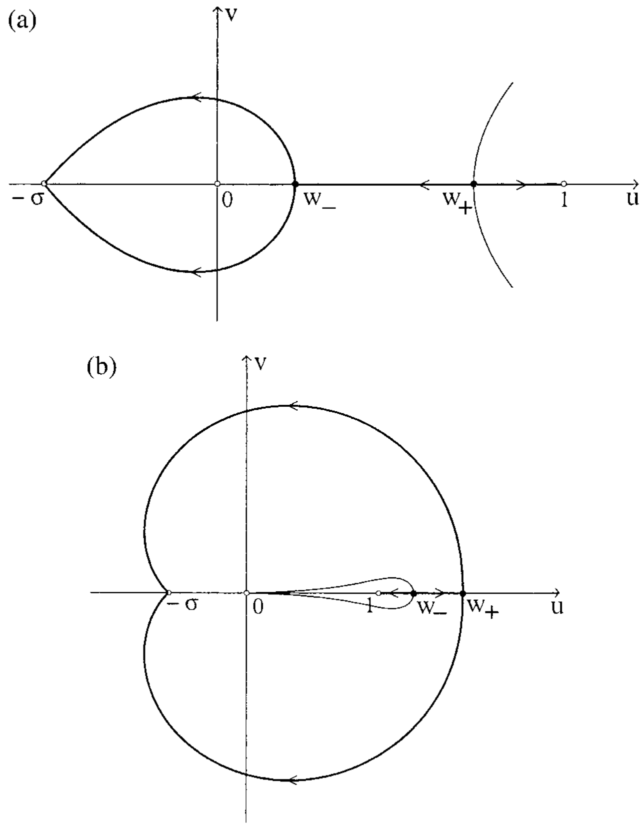
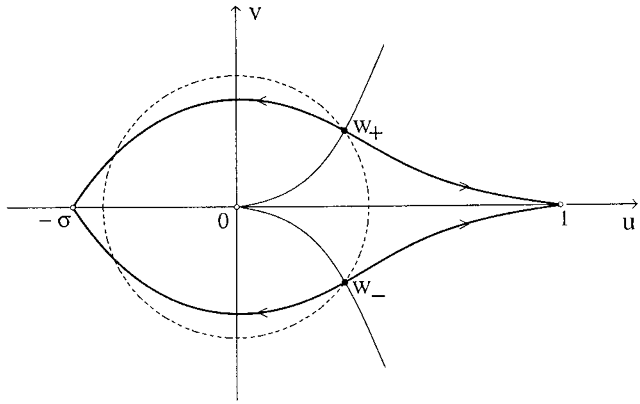
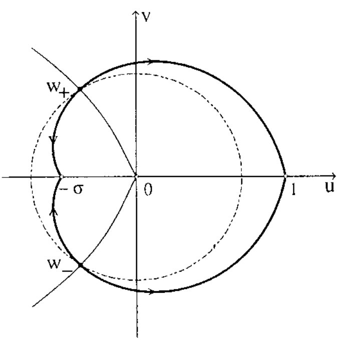


FIG. 2. (a) Steepest descent paths in the w-plane ($\lambda_+ < \lambda < 1$ and $v < q$). (b) Steepest descent paths in the w-plane ($\lambda_+ < \lambda < 1$ and $v > q$). (c) Steepest descent paths in the w-plane ($\lambda_- < \lambda < \lambda_+$). (d) Steepest descent paths in the w-plane ($0 < \lambda < \lambda_-$ and $v < p$). (e) Steepest descent paths in the w-plane ($0 < \lambda < \lambda_-$ and $v > p$).

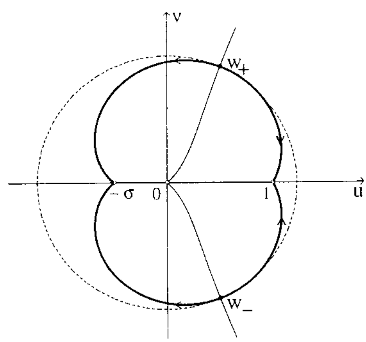
(c)



$(\lambda_- < \lambda < \lambda_+, \nu < p < q)$



$(\lambda_- < \lambda < \lambda_+, p < \nu < q)$



$(\lambda_- < \lambda < \lambda_+, p < q < \nu)$

(d)

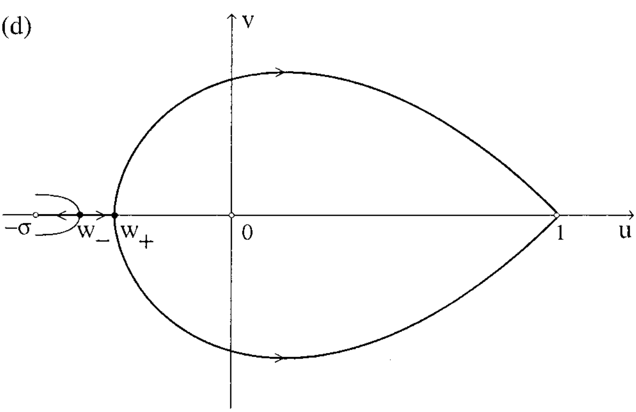


FIG. 2. *Continued*

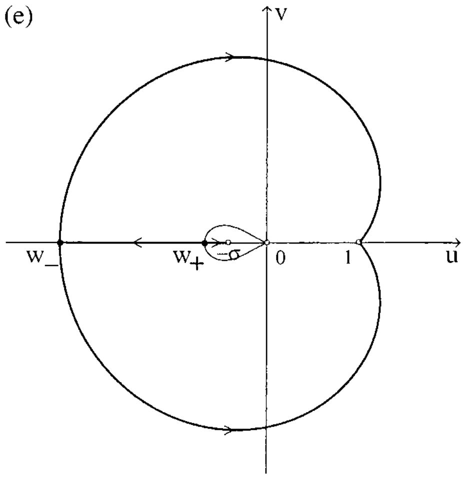


FIG. 2. *Continued*

Case (e). $0<\lambda<\lambda_-$ and $\nu>p$. In this case, $w_-<w_+<-\sigma<0$. Note that there is a cut along the infinite half-line $(-\infty, -\sigma)$. By (2.9), we have

$$\operatorname{Im} F(w_\pm^+) = \lambda \arg(\sigma + w_\pm^+) - \nu \arg w_\pm = \lambda\pi - \nu\pi$$

and

$$\operatorname{Im} F(w_\pm^-) = \lambda \arg(\sigma + w_\pm^-) - \nu \arg w_\pm = -\lambda\pi - \nu\pi.$$

The steepest descent paths are given by

$$(1-\lambda)\arg(1-w) + \lambda \arg(\sigma + w) - \nu \arg w = \begin{cases} -\pi(\nu-\lambda) & \text{if} \quad v>0, \\ -\pi(\nu+\lambda) & \text{if} \quad v<0, \end{cases}$$

and points in the interval $(-\infty, -\sigma)$ satisfy this equation; see Fig. 2e.

4. RELATION TO THE PARABOLIC CYLINDER FUNCTION

Since the phase function $F(w, \lambda)$ in (2.6) has two saddle points $w_+(\lambda)$ and $w_-(\lambda)$ and these two points coalesce when $\lambda=\lambda_+$ and $\lambda=\lambda_-$, our present situation is very much like those in the cases of Meixner [6] and Meixner–Pollaczek [8] polynomials. Thus, we should compare Krawtchouk polynomials with the parabolic cylinder function given by

$$U(d, x) = \frac{\Gamma(1/2-d)}{2\pi i}\, e^{-x^2/4} \int_{-\infty}^{(0^+)} e^{xz - z^2/2} z^{d-1/2}\, dz; \tag{4.1}$$

see [1, p. 687]. Letting $x=\sqrt{n}\,\beta$ and $d=-n-1/2$, we have

$$U\left(-n-\frac{1}{2}, \sqrt{n}\,\beta\right)=\frac{n!}{2\pi i}\,e^{-n\beta^2/4}n^{-n/2}\int_{-\infty}^{(0+)} e^{n\Psi(z,\,\beta)}\,\frac{dz}{z}, \tag{4.2}$$

where

$$\Psi(z, \beta)=-\log z+\beta z-\frac{z^2}{2}. \tag{4.3}$$

The saddle points of $\Psi(z, \beta)$ are given by

$$z_{\pm}=\frac{\beta\pm\sqrt{\beta^2-4}}{2}, \tag{4.4}$$

and they coincide when $\beta=\pm 2$. The movements of z_+ and z_- have been discussed in [6], and they are found to be similar to those of w_+ and w_-. Near $w=0$, $F(w, \lambda)$ has the approximation

$$F(w, \lambda)=-v\log w+\left(\frac{\lambda}{\sigma}-1+\lambda\right)w-\left(\frac{\lambda}{2\sigma^2}+\frac{1-\lambda}{2}\right)w^2+\lambda\log\sigma+\cdots. \tag{4.5}$$

Comparing (4.3) and (4.5) suggests that we define the mapping $w\to z(w)$ by setting

$$v^{-1}F(w, \lambda)=\Psi(z, \beta)+\gamma, \tag{4.6}$$

where γ is a constant. Clearly, $w=0$ is mapped into $z=0$, regardless of the value of γ. For the transformation (4.6) to be analytic in our region of interest, we require $dw/dz\neq 0$ or ∞. Since

$$v^{-1}F_w(w, \lambda)\,\frac{dw}{dz}=-\frac{1}{z}(z-z_+)(z-z_-) \tag{4.7}$$

and $F_w(w, \lambda)$ vanishes at $w=w_{\pm}$, the points $w_{\pm}$ must in some way correspond to the points $z_{\pm}$. If we assign $z(w_+)=z_+$ and $z(w_-)=z_-$, then we have the following system of two equations and two unknowns,

$$v^{-1}F(w_+, \lambda)=\Psi(z_+, \beta)+\gamma, \tag{4.8}$$

$$v^{-1}F(w_-, \lambda)=\Psi(z_-, \beta)+\gamma. \tag{4.9}$$

(In some cases, we shall use the correspondence $w_{\mp}^{\pm}\leftrightarrow z_{\pm}$ or $w_{+}^{\pm}\leftrightarrow z_{\pm}$.) The existence of a unique solution (β, γ) to the above system of nonlinear equations can be established by using an argument similar to those used in [6, Theorems 1 and 2; 8, Lemma 1]. Hence, our presentation here will be brief.

Subtracting (4.9) from (4.8) gives

$$v^{-1}[F(w_+, \lambda) - F(w_-, \lambda)] = \Psi(z_+, \beta) - \Psi(z_-, \beta). \tag{4.10}$$

To solve the system (4.8)–(4.9) is equivalent to solve the system (4.9)–(4.10). Thus we only need to show that for each value of $\lambda \in (0, 1)$, there is a value $\beta \in (-\infty, \infty)$ such that (4.10) holds, since substituting (4.9) in (4.10) will give a unique value for γ. Put

$$f(\lambda) \equiv F(w_+, \lambda) - F(w_-, \lambda) \tag{4.11}$$

and

$$\psi(\beta) = \Psi(z_+, \beta) - \Psi(z_-, \beta). \tag{4.12}$$

The graph of $\psi(\beta)$ is shown in Fig. 3; cf. [6, Lemma, p. 125]. To draw the graph of $f(\lambda)$, there are five cases to be considered, namely, (i) $v < p < q$, (ii) $p = v < q$, (iii) $p < v < q$, (vi) $p < v = q$ and (v) $p < q < v$. However, here we present only the discussions for the first and the last cases. The other cases can be handled in similar manners.

Case (i). From (4.11) and (2.6), it follows that

$$\begin{aligned} f(\lambda) = {} & (1-\lambda)[\log(1-w_+) - \log(1-w_-)] \\ & + \lambda[\log(\sigma + w_+) - \log(\sigma + w_-)] - v[\log w_+ - \log w_-]. \end{aligned} \tag{4.13}$$

When $\lambda_+ < \lambda < 1$ and $v < q$, the saddle points w_+ and w_- are real and are arranged in the order

$$0 < w_-(1) < w_- < r_0 < w_+ < w_+(1),$$

where $w_-(1) = \sigma v/(1-v)$ and $w_+(1) = 1$; see Section 2 for the movements of w_+ and w_-. Thus, it is clear that $\operatorname{Im} f(\lambda) = 0$ for $\lambda \in (\lambda_+, 1)$. Note that w_+ and w_- depend on λ. Hence

$$f'(\lambda) = F_w(w_+, \lambda)\, w'_+(\lambda) + F_\lambda(w_+, \lambda) - F_w(w_-, \lambda)\, w'_-(\lambda) - F_\lambda(w_-, \lambda).$$

Since $w_\pm$ are the saddle points of $F(w, \lambda)$, the last equation gives

$$\begin{aligned} \frac{d}{d\lambda} \operatorname{Re} f(\lambda) &= \operatorname{Re} f'(\lambda) = \operatorname{Re}[F_\lambda(w_+, \lambda) - F_\lambda(w_-, \lambda)] \\ &= \log \frac{1-w_-}{1-w_+} + \log \frac{\sigma + w_+}{\sigma + w_-}. \end{aligned} \tag{4.14}$$

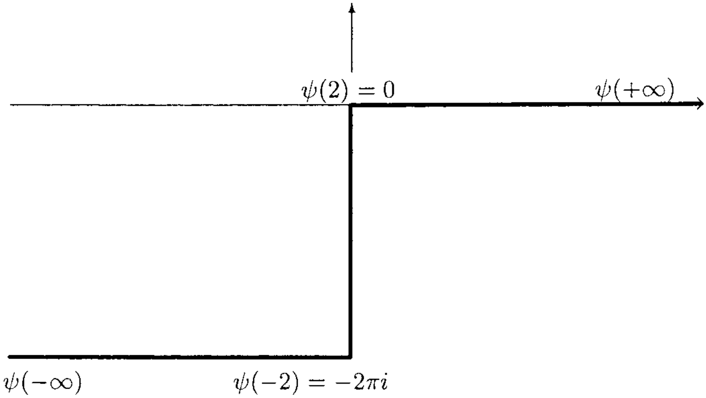


FIG. 3. Graph of $\psi(\beta)$ in the complex ψ-plane.

The arguments of the two logarithms being greater than 1 implies that $\operatorname{Re} f'(\lambda) > 0$, i.e., $\operatorname{Re} f(\lambda)$ is strictly increasing in $\lambda_+ < \lambda < 1$. To find the limit of $\operatorname{Re} f(\lambda)$ as $\lambda \to 1^-$, we first note that

$$(1-w_+)(1-w_-) = 1-(w_+ + w_-) + w_+ w_-,$$

and that by (2.12)

$$(1-w_+)(1-w_-) = 1 - \frac{\lambda(1+\sigma)-\sigma-v+\sigma v}{1-v} + \frac{\sigma v}{1-v} = \frac{1+\sigma}{1-v}(1-\lambda).$$

As $\lambda \to 1^-$, $w_-(\lambda) \to w_-(1^-) = \sigma v/(1-v)$. Since $v < p < q$, it follows that

$$1 - w_-(1^-) = \frac{1-v(1+\sigma)}{1-v} > 0$$

and

$$1 - w_+ \sim \frac{1+\sigma}{1-v(1+\sigma)}(1-\lambda).$$

Therefore, as $\lambda \to 1^-$,

$$(1-\lambda)\log(1-w_+) \sim (1-\lambda)\log(1-\lambda) \to 0$$

and

$$\operatorname{Re} f(\lambda) \to \log\frac{(1+\sigma)(1-v)}{\sigma} + v\log\frac{\sigma v}{1-v} \equiv \operatorname{Re} f(1^-). \tag{4.15}$$

On the other hand, since $w_\pm \to r_0$ as $\lambda \to (\lambda_+)^+$, we have immediately

$$\mathrm{Re}\, f(\lambda) \to \mathrm{Re}\, f(\lambda_+) = 0; \tag{4.16}$$

cf. Fig. 1a.

When $\lambda_- < \lambda < \lambda_+$, w_+ and w_- are complex conjugates. Hence, $F(w_-, \lambda) = \overline{F(w_+, \lambda)}$ and $f(\lambda) = 2i\, \mathrm{Im}\, F(w_+, \lambda)$. As a result, we have $\mathrm{Re}\, f(\lambda) = 0$ and

$$\mathrm{Im}\, f(\lambda) = 2[(1-\lambda) \arg(1-w_+) + \lambda \arg(\sigma + w_+) - v \arg w_+]. \tag{4.17}$$

Again since w_+ is a saddle point,

$$\frac{d}{d\lambda} \mathrm{Im}\, f(\lambda) = 2\, \mathrm{Im}\, F_\lambda(w_+, \lambda) = 2[-\arg(1-w_+) + \arg(\sigma + w_+)].$$

It is readily seen that $\arg(1-w_+) < 0$ and $\arg(\sigma + w_+) > 0$. Therefore, $\mathrm{Im}\, f(\lambda)$ is strictly increasing in $\lambda_- < \lambda < \lambda_+$. As $\lambda \to (\lambda_+)^-$, we have $w_+ \to r_0$ and $\mathrm{Im}\, f(\lambda) \to \mathrm{Im}\, f(\lambda_+) = 0$ by virtue of (4.17) and (2.21). As $\lambda \to (\lambda_-)^+$, we have $w_+ \to r_0 e^{i\pi}$ and $\mathrm{Im}\, f(\lambda) \to \mathrm{Im}\, f(\lambda_-) = -2v\pi$ by (2.22).

When $0 < \lambda < \lambda_-$ and $v < p$, the saddle points are real and negative. They are arranged in the order

$$w_-(0) < w_- < -r_0 < w_+ < w_+(0) < 0,$$

where $w_-(0) = -\sigma$ and $w_+(0) = -v/(1-v)$. Thus, from (4.13) we obtain

$$\mathrm{Re}\, f(\lambda) = (1-\lambda)[\log(1-w_+) - \log(1-w_-)] \\ + \lambda[\log(\sigma + w_+) - \log(\sigma + w_-)] - v[\log(-w_+) - \log(-w_-)] \tag{4.18}$$

and $\mathrm{Im}\, f(\lambda) = -2\pi v$; see Fig. 1a. Differentiation of (4.18) gives

$$\mathrm{Re}\, f'(\lambda) = \log \frac{1-w_-}{1-w_+} + \log \frac{\sigma + w_+}{\sigma + w_-} > 0.$$

Hence, $\mathrm{Re}\, f(\lambda)$ is strictly increasing in $0 < \lambda < \lambda_-$. As $\lambda \to (\lambda_-)^-$, we have $w_\pm \to -r_0$, and from (4.18) it follows that $\mathrm{Re}\, f(\lambda) \to \mathrm{Re}\, f(\lambda_-) = 0$. To find the limit of $\mathrm{Re}\, f(\lambda)$ as $\lambda \to 0^+$, we note that

$$(\sigma + w_+)(\sigma + w_-) = \sigma^2 + \sigma(w_+ + w_-) + w_+ w_-$$

$$= \sigma^2 + \sigma \frac{\lambda(1+\sigma) - \sigma - v + \sigma v}{1-v} + \frac{\sigma v}{1-v}$$

$$= \frac{\sigma(1+\sigma)}{1-v} \lambda \tag{4.19}$$

on account of (2.12). Since $v < p < q$, (2.16) gives

$$\sigma + w_+(0) = \sigma - \frac{v}{1-v} = \frac{\sigma - v(1+\sigma)}{1-v} > 0$$

and (4.19) gives

$$\sigma + w_- \sim \frac{\sigma(1+\sigma)}{\sigma - v(1+\sigma)} \lambda.$$

Thus, as $\lambda \to 0^+$,

$$\lambda \log(\sigma + w_-) \sim \lambda \log \lambda \to 0$$

and

$$\operatorname{Re} f(\lambda) \to \left[\log \left(1 + \frac{v}{1-v} \right) - \log(1+\sigma) \right] - v \left[\log \frac{v}{1-v} - \log \sigma \right] \equiv \operatorname{Re} f(0).$$

In summary, the graph of the function $f(\lambda)$ when $v < p < q$ is as shown in Fig. 4.

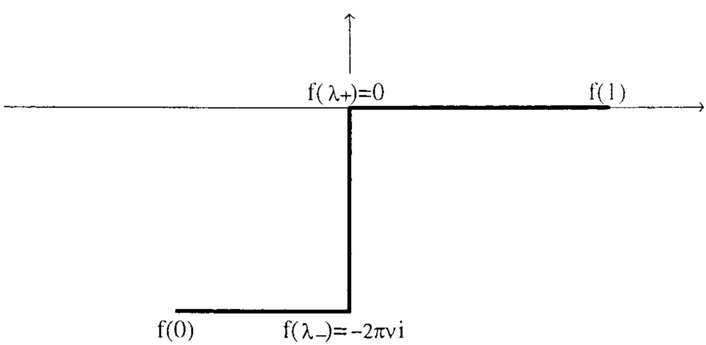


FIG. 4. Graph of $f(\lambda)$ in the complex f-plane ($v < p < q$).

Case (v). When $\nu > q$ and $\lambda_+ < \lambda < 1$, the saddle points w_+ and w_- are real, positive and arranged in the order

$$w_-(1) < w_- < r_0 < w_+ < w_+(1),$$

where $w_-(1) = 1$ and $w_+(1) = \sigma\nu/(1-\nu)$; see Fig. 1a. Instead of (4.8)–(4.9), we shall use the correspondence $w_\mp^\pm \leftrightarrow z_\pm$; that is, we shall set

$$\nu^{-1} F(w_-^+, \lambda) = \Psi(z_+, \beta) + \gamma, \tag{4.20}$$

$$\nu^{-1} F(w_-^-, \lambda) = \Psi(z_-, \beta) + \gamma. \tag{4.21}$$

The function $f(\lambda) = F(w_-^+, \gamma) - F(w_-^-, \gamma)$ becomes

$$f(\lambda) = (1-\lambda)[\log(1-w_-^+) - \log(1-w_-^-)]$$
$$+ \lambda[\log(\sigma + w_-^+) - \log(\sigma + w_-^-)] - \nu[\log w_-^+ - \log w_-^-].$$

Clearly, $\operatorname{Re} f(\lambda) = 0$, $\operatorname{Im} f(\lambda) = -2\pi(1-\lambda)$, and $\operatorname{Im} f(\lambda)$ is strictly increasing from $-2\pi(1-\lambda_+)$ to 0 as λ increases from λ_+ to 1.

When $\lambda_- < \lambda < \lambda_+$, w_+ and w_- are complex conjugates as in case (i), and the argument there can be used to show that $\operatorname{Re} f(\lambda) = 0$ and $\operatorname{Im} f(\lambda)$ is strictly increasing. By (2.21), $r_0 > 1$ in the present case. Hence it follows from (4.17) that $\operatorname{Im} f(\lambda) \to \operatorname{Im} f(\lambda_+) = -2\pi(1-\lambda_+)$ as $\lambda \to (\lambda_+)^-$ and $\operatorname{Im} f(\lambda) \to \operatorname{Im} f(\lambda_-) = -2\pi(\nu - \lambda_-)$ as $\lambda \to (\lambda_-)^+$; cf. Fig. 1a.

When $\nu > p$ and $0 < \lambda < \lambda_-$, the saddle points w_+ and w_- are real and negative. They are arranged in the order

$$w_-(0) < w_- < -r_0 < w_+ < w_+(0)$$

with $w_-(0) = -\nu/(1-\nu)$ and $w_+(0) = -\sigma$; cf. (2.22). In this case, we shall use the correspondence $w_+^\pm \leftrightarrow z_\pm$; that is, we shall set

$$\nu^{-1} F(w_+^+, \lambda) = \Psi(z_+, \beta) + \gamma, \tag{4.22}$$

$$\nu^{-1} F(w_+^-, \lambda) = \Psi(z_-, \beta) + \gamma, \tag{4.23}$$

and $f(\lambda) = F(w_+^+, \lambda) - F(w_+^-, \lambda)$. From (2.6), we have

$$f(\lambda) = (1-\lambda)[\log(1-w_+^+) - \log(1-w_+^-)]$$
$$+ \lambda[\log(\sigma + w_+^+) - \log(\sigma + w_+^-)] - \nu[\log w_+^+ - \log w_+^-].$$

Clearly, $\operatorname{Re} f(\lambda) = 0$. Since $w_+ < -\sigma < 0$, it is readily seen that $\operatorname{Im} f(\lambda) = -2\pi(\nu - \lambda)$. Thus, $\operatorname{Im} f(\lambda)$ is strictly increasing in $0 < \lambda < \lambda_-$ with $\operatorname{Im} f(0) = -2\pi\nu$ and $\operatorname{Im} f(\lambda_-) = -2\pi(\nu - \lambda_-)$. In summary, the graph of $f(\lambda)$ in the case $p < q < \nu$ is as shown in Fig. 5.

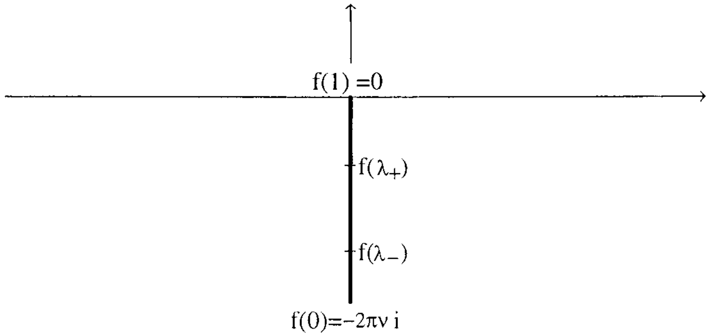


FIG. 5. Graph of f in the complex f-plane ($p<q<\nu$).

In all cases (i)–(v), we have found that either Re $f=0$ and Im f is strictly increasing or Im $f=0$ and Re f is strictly increasing. Furthermore, the range of $\nu^{-1}f(\lambda)$ is always contained in the range of $\psi(\beta)$; compare the graphs in Figs. 3–5. Thus it follows that for each $\lambda \in (0, 1)$, there exists a value $\beta \in (-\infty, \infty)$ such that

$$\nu^{-1}f(\lambda)=\psi(\beta). \tag{4.24}$$

That is, for each $\lambda \in (0, 1)$ and $\nu \in (0, 1)$, we have proved that there exists a solution (β, γ) to the nonlinear system (4.8)–(4.9), (4.20)–(4.21), or (4.22)–(4.23), depending on the values of λ and ν. With the values of β and γ so chosen, the transformation $w \leftrightarrow z$ defined in (4.6) can be shown, as in many previous papers [2, 3, 6, 8], to be one-to-one and analytic along the whole steepest descent path through relevant saddle points of the integral in (2.5). The shapes of this steepest descent path, denoted by Γ, and their images in the z-plane, denoted by Γ', are depicted in Figs. 6a–6e.

Returning to (2.5), we first deform the small circular contour C into the steepest descent path Γ, and then make the change of variable from w to z. The image of Γ is the steepest descent path of $\Psi(z, \beta)$, and the integral in (2.5) becomes

$$K_n(\lambda N)=\frac{p^n\sigma^{-\lambda N}}{2\pi i}\,e^{n\gamma}\int_{\Gamma'} e^{n\Psi(z,\beta)}\,\frac{w'(z)}{w(z)}\,dz. \tag{4.25}$$

The contour Γ' in Figs. 6a, 6b, and 6c can be deformed into an infinite loop beginning and ending at $-\infty$, so that the integral in (4.25) becomes

$$K_n(\lambda N)=\frac{p^n\sigma^{-\lambda N}}{2\pi i}\,e^{n\gamma}\int_{-\infty}^{(0^+)} \Phi(z)\,e^{n\Psi(z,\beta)}\,\frac{dz}{z}, \tag{4.26}$$

where

$$\Phi(z) = z\,\frac{w'(z)}{w(z)}. \tag{4.27}$$

However, the contour Γ' in Figs. 6d and 6e can be deformed only into an infinite loop beginning and ending at $+\infty$; that is,

$$K_n(\lambda N) = \frac{p^n \sigma^{-\lambda N}}{2\pi i}\, e^{n\gamma} \int_{+\infty}^{(0^+)} \Phi(z)\, e^{n\Psi(z,\,\beta)}\, \frac{dz}{z}. \tag{4.28}$$

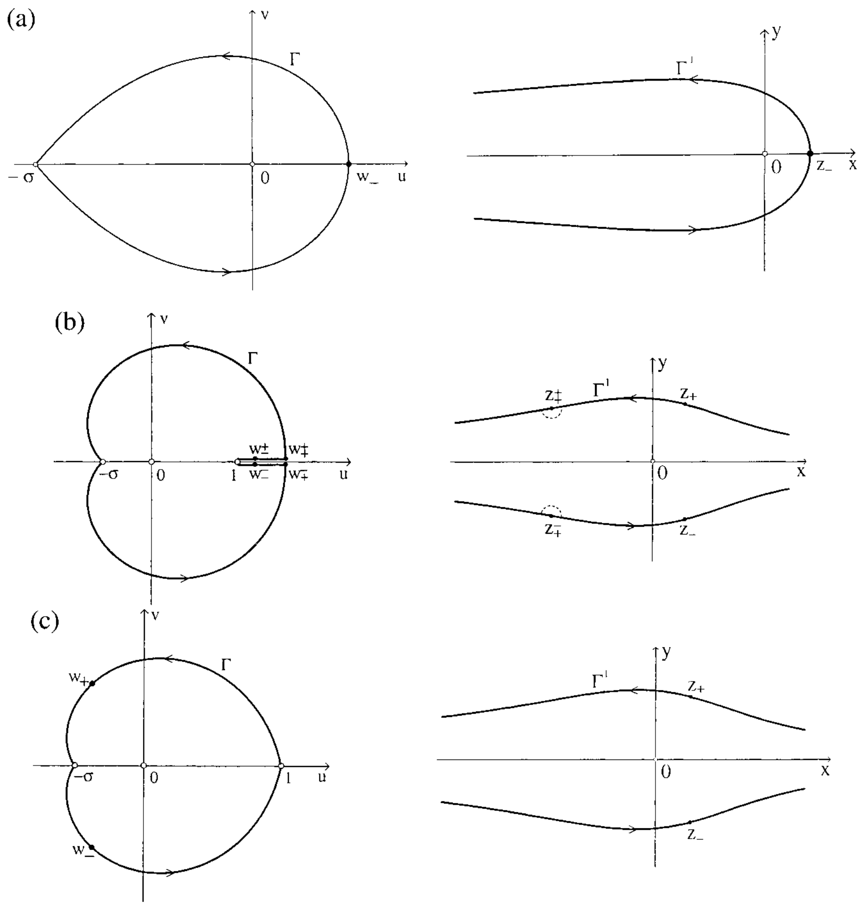


FIG. 6. (a) Contours Γ and Γ' ($\lambda_+ < \lambda < 1$ and $\nu < q$). (b) Contours Γ and Γ' ($\lambda_+ < \lambda < 1$ and $\nu > q$). (c) Contours Γ and Γ' ($\lambda_- < \lambda < \lambda_+$). (d) Contours Γ and Γ' ($0 < \lambda < \lambda_-$ and $\nu > p$). (e) Contours Γ and Γ' ($0 < \lambda < \lambda_-$ and $\nu < p$).

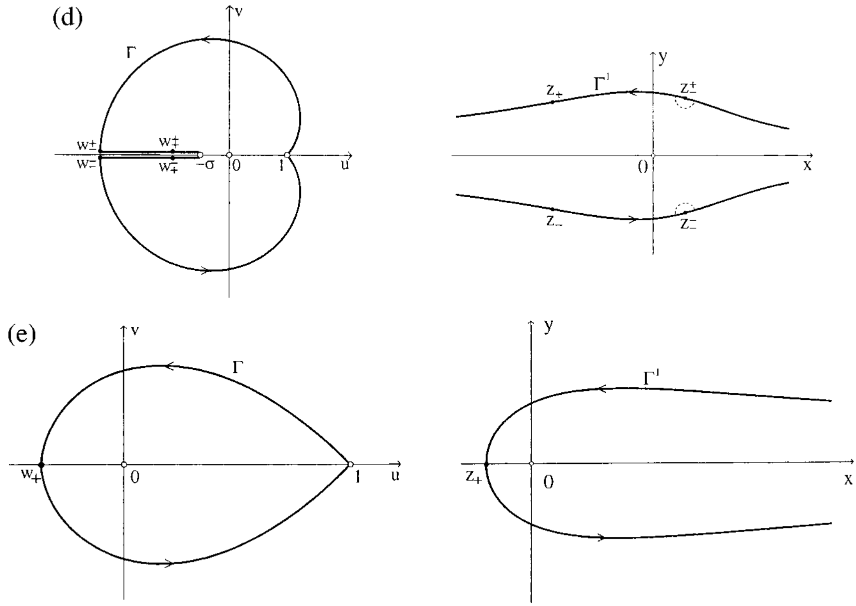


FIG. 6. *Continued*

We shall see in the following section that these two seemingly different integrals will lead to the same uniform asymptotic expansion.

5. ASYMPTOTIC EXPANSION

Before proceeding to the derivation of the uniform asymptotic expansion, we wish to first point out that the points $z^{\pm}_{+}$ in Fig. 6b (i.e., the images of $w^{\pm}_{+}$) are singular points of the integrand in (4.25). Similarly, the points $z^{\pm}_{-}$ in Fig. 6d (i.e., the images of $w^{\pm}_{-}$) are also singular points of this integrand. For this reason, we have used small semicircles to indicate that there are indentations at the points $z^{\pm}_{+}$ and $z^{\pm}_{-}$ on the path Γ' in these figures. However, the integral in (4.25) does exist at these points; that is, $z^{\pm}_{+}$ and $z^{\pm}_{-}$ are integrable singularities. To see all these, we return to (4.6) and write

$$w'(z) = \nu \frac{\Psi_z(z, \beta)}{F_w(z, \lambda)}. \tag{5.1}$$

From (2.6), we have

$$F_w(w, \lambda) = \frac{(1-\nu)(w-w_+)(w-w_-)}{w(w-1)(w+\sigma)}. \tag{5.2}$$

Coupling (4.7) and (5.2) gives

$$w'(z)=\frac{v}{1-v}(1-w)(\sigma+w)\,\frac{w(z-z_+)(z-z_-)}{z(w-w_+)(w-w_-)}. \tag{5.3}$$

Substituting (5.3) in (4.27) yields

$$\Phi(z)=\frac{v}{1-v}(1-w)(\sigma+w)\,\frac{(z-z_+)(z-z_-)}{(w-w_+)(w-w_-)}. \tag{5.4}$$

Under the correspondence $w^{\pm}_{-}\leftrightarrow z_{\pm}$, the saddle points $w^{\pm}_{+}$ of $F(w,\lambda)$ correspond to the points $z^{\pm}_{+}$ which are not saddle points of $\Psi(z,\beta)$; see Fig. 6b. That is, $F_w(w^{\pm}_{+},\lambda)=0$ but $\Psi_z(z^{\pm}_{+},\beta)\neq 0$. Analogously, under the correspondence $w^{\pm}_{+}\leftrightarrow z_{\pm}$, $F_w(w^{\pm}_{-},\lambda)=0$ but $\Psi_z(z^{\pm}_{-},\beta)\neq 0$; see Fig. 6d. Since the two cases are similar, it suffices to discuss just the case $w^{\pm}_{-}\leftrightarrow z_{\pm}$. When $z\to z_{\pm}$, we have $w\to w^{\pm}_{-}$ and

$$\lim_{z\to z_{\pm}}\frac{w-w^{\pm}_{-}}{z-z_{\pm}}=w'(z_{\pm}).$$

But, when $z\to z^{\pm}_{+}$, we have $w\to w^{\pm}_{+}$ and hence $w-w_+\to w^{\pm}_{+}-w_+=0$ but $z-z_+\to z^{\pm}_{+}-z_+\neq 0$. Thus,

$$\lim_{z\to z^{\pm}_{+}}\frac{z-z_+}{w-w_+}=\infty$$

and $z^{\pm}_{+}$ are two singular points of $\Phi(z)$. Put $W=F(w,\lambda)$, $Z=\Psi(z,\beta)$,

$$F(w^{\pm}_{+},\lambda)\equiv W^{\pm}_{+}\qquad\text{and}\qquad \Psi(z^{\pm}_{+},\beta)\equiv Z^{\pm}_{+}.$$

On one hand, by Taylor's expansion,

$$W-W^{\pm}_{+}\sim\tfrac{1}{2}F_{ww}(w^{\pm}_{+},\lambda)(w-w^{\pm}_{+})^2$$

which in turn gives

$$w-w^{\pm}_{+}\sim\left[\frac{2}{F_{ww}(w^{\pm}_{+},\lambda)}(W-W^{\pm}_{+})\right]^{1/2}. \tag{5.5}$$

On the other hand,

$$Z-Z^{\pm}_{+}\sim\Psi_z(z^{\pm}_{+},\beta)(z-z^{\pm}_{+}). \tag{5.6}$$

From (4.6), $v^{-1}W = Z + \gamma$, $v^{-1}W^{\pm}_{+} = Z^{\pm}_{+} + \gamma$ and it follows that $W - W^{\pm}_{+} = v[Z - Z^{\pm}_{+}]$. Coupling (5.5) and (5.6) yields

$$w - w^{\pm}_{+} \sim \sqrt{\frac{2v\Psi_z(z^{\pm}_{+}, \beta)}{F_{ww}(w^{\pm}_{+}, \lambda)}}\,(z - z^{\pm}_{+})^{1/2}, \qquad \text{as} \qquad z \to z^{\pm}_{+},$$

and

$$\Phi(z) = O((z - z^{\pm}_{+})^{-1/2}), \qquad \text{as} \qquad z \to z^{\pm}_{+}. \tag{5.7}$$

Therefore the integral in (4.25) is convergent, except that one should first make a semi-circular indentation at the points $z^{\pm}_{+}$ on the steepest descent path Γ' and then let the radii of these semi-circles tend to zero.

Now we recall the integral representation (4.2)

$$U\left(-n - \frac{1}{2}, \sqrt{n}\,\beta\right) = \frac{n!}{2\pi i}\, e^{-n\beta^2/4} n^{-n/2} \int_{-\infty}^{(0+)} e^{n\Psi(z, \beta)}\,\frac{dz}{z},$$

where $\Psi(z, \beta) = -\log z + \beta z - \frac{1}{2} z^2$. For simplicity, we define the new function

$$V_n(x) = e^{x^2/4} U(-n - \tfrac{1}{2}, x). \tag{5.8}$$

Clearly

$$V_n(\beta\sqrt{n}) = \frac{n!}{2\pi i}\, n^{-n/2} \int_{-\infty}^{(0+)} e^{n\Psi(z, \beta)}\,\frac{dz}{z}, \tag{5.9}$$

and from (4.1)

$$V_n'(\beta\sqrt{n}) = \frac{n!}{2\pi i}\, n^{(1-n)/2} \int_{-\infty}^{(0+)} e^{n\Psi(z, \beta)}\,dz. \tag{5.10}$$

Next we return to the integral in (4.26), and put $\Phi_0(z) \equiv \Phi(z)$. Define recursively

$$\begin{cases} \Phi_l(z) = a_l + b_l z + (z - z_+)(z - z_-)\,\Theta_l(z), \\ \Phi_{l+1}(z) = z\Theta_l'(z), \end{cases} \qquad l = 0, 1, 2, ..., \tag{5.11}$$

with

$$a_l = \frac{z_+ \Phi_l(z_-) - z_- \Phi_l(z_+)}{z_+ - z_-} \tag{5.12}$$

and

$$b_l = \frac{\Phi_l(z_+) - \Phi_l(z_-)}{z_+ - z_-}. \tag{5.13}$$

By repeated integration by parts, we obtain the expansion

$$K_n(\lambda N) = \frac{1}{n!} p^n \sigma^{-\lambda N} n^{n/2} e^{\gamma n} \times \left\{ V_n(\beta\sqrt{n}) \sum_{l=0}^{m-1} \frac{a_l}{n^l} + \frac{1}{\sqrt{n}} V_n'(\beta\sqrt{n}) \sum_{l=0}^{m-1} \frac{b_l}{n^l} + \varepsilon_m \right\}, \tag{5.14}$$

where

$$\varepsilon_m = \frac{n!}{2\pi i} n^{-n/2-m} \int_{-\infty}^{(0+)} \Phi_m(z)\, e^{n\Psi(z,\beta)} \frac{dz}{z}. \tag{5.15}$$

Since the derivation of expansion (5.14) is exactly the same as that in [6, (3.25)] and [6, (3.14)], it will not be repeated here.

Regarding the integral in (4.28), we first note that $\Psi(-z, \beta) = \Psi(z, -\beta) - i\pi$. Hence

$$\int_{+\infty}^{(0+)} e^{n\Psi(z,\beta)} \frac{dz}{z} = e^{-in\pi} \int_{-\infty}^{(0+)} e^{n\Psi(z,-\beta)} \frac{dz}{z}$$
$$= \frac{2\pi i}{n!} n^{n/2} (-1)^n V_n(-\beta\sqrt{n}).$$

Since $U(-n-\frac{1}{2}, -x) = (-1)^n U(-n-\frac{1}{2}, x)$, we have $V_n(-\beta\sqrt{n}) = (-1)^n V_n(\beta\sqrt{n})$ and

$$\int_{+\infty}^{(0+)} e^{n\Psi(z,\beta)} \frac{dz}{z} = \frac{2\pi i}{n!} n^{n/2} V_n(\beta\sqrt{n}). \tag{5.16}$$

Similarly, by using (5.10), we also have

$$\int_{+\infty}^{(0+)} e^{n\Psi(z,\beta)}\, dz = \frac{2\pi i}{n!} n^{n/2} \frac{1}{\sqrt{n}} V_n'(\beta\sqrt{n}). \tag{5.17}$$

A comparison of (5.9)–(5.10) with (5.16)–(5.17) reveals that the expansion (5.14) actually holds for all $\lambda \in (0, 1)$, despite the fact that the loop contour in (4.28) begins and ends at $+\infty$ when $0 < \lambda < \lambda_-$. (However, in this case, the contour in the error term remains to be of the form $\int_{+\infty}^{(0+)}$.)

The integral in (5.15) can be estimated as in [6], and the result is that for arbitrarily small $\varepsilon > 0$ and $\delta > 0$, there are constants P_m and Q_m, which are independent of n, $\lambda \in [\varepsilon, 1-\varepsilon]$ and $v \in [\delta, 1-\delta]$, such that

$$|\varepsilon_m| \leqslant \frac{P_m}{n^m} |V_n(\beta\sqrt{n})| + \frac{Q_m}{n^{m+1/2}} |V_n'(\beta\sqrt{n})|. \tag{5.18}$$

This, of course, establishes (5.14) as an asymptotic expansion, uniformly valid for λ and v in any compact subsets of (0, 1).

6. NON-UNIFORM APPROXIMATIONS

Taking the dominant term in the expansion (5.14), we have

$$K_n(\lambda N) \sim \frac{1}{n!} p^n \sigma^{-\lambda N} n^{n/2} e^{\gamma n} \left\{ a_0 V_n(\beta\sqrt{n}) + \frac{b_0}{\sqrt{n}} V_n'(\beta\sqrt{n}) \right\}, \tag{6.1}$$

as $n \to \infty$, uniformly for $\lambda \in [\varepsilon, 1-\varepsilon]$ and $v \in [\delta, 1-\delta]$, where ε and δ are arbitrarily small but fixed positive numbers. Despite its simplicity, in practice we need to express the coefficients a_0, b_0 and the parameters β, γ in terms of the original variables λ and v. This is possible, only if we are willing to restrict λ and v to smaller regions. In this section, we consider only the following four special cases. Other cases can be dealt with in similar manners.

(i) $\lambda_- < \lambda < \lambda_+$, $\delta \leqslant v \leqslant 1-\delta$ and $\lambda \not\to \lambda_\pm$.

(ii) $\lambda_+ < \lambda \leqslant 1-\varepsilon$, $\delta \leqslant v < q$ and $\lambda \not\to \lambda_+$; or $\varepsilon \leqslant \lambda < \lambda_-$, $\delta \leqslant v < p$ and $\lambda \not\to \lambda_-$.

(iii) $\lambda_+ < \lambda \leqslant 1-\varepsilon$, $q < v \leqslant 1-\delta$ and $\lambda \not\to \lambda_+$; or $\varepsilon \leqslant \lambda < \lambda_-$, $p < v \leqslant 1-\delta$ and $\lambda \not\to \lambda_-$.

(iv) $\lambda \to \lambda_+$, $n^{2/3}(\lambda - \lambda_+) = O(1)$ and $\delta \leqslant v < q$; or $\lambda \to \lambda_-$, $n^{2/3}(\lambda - \lambda_-) = O(1)$ and $\delta \leqslant v < p$.

For each of these cases, we shall assume that $0 < p < q < 1$; see (2.10). Results for the corresponding cases when $p = q$ can be obtained by simply taking $p = q = \frac{1}{2}$, i.e., $\sigma = 1$, in the anticipated results of the four cases listed above. We also note that in cases (ii) and (iii) there is no need to consider $v = q$ or $v = p$, since $\lambda_+ = 1$ when $v = q$ and $\lambda_- = 0$ when $v = p$.

Case (i). When $\lambda_- < \lambda < \lambda_+$, both $w_\pm$ and $z_\pm$ are complex conjugates. From [11, pp. 143 and 145], we have

$$V_n(\beta\sqrt{n}) \sim 2e^{n(\beta^2/4 - 1/2)} n^{n/2} (4-\beta^2)^{-1/4} \cos\psi_n \tag{6.2}$$

and

$$\frac{1}{\sqrt{n}} V_n'(\beta\sqrt{n}) \sim 2e^{n(\beta^2/4-1/2)} n^{n/2} (4-\beta^2)^{-1/4} \cos(\psi_n + \varphi_+), \tag{6.3}$$

where

$$\psi_n = n\left(\frac{\beta}{4}\sqrt{4-\beta^2} - \varphi_+\right) - \frac{1}{2}\varphi_+ + \frac{\pi}{4} \tag{6.4}$$

and $z_+ = e^{i\varphi_+}$, $0 < \varphi_+ < \pi$; i.e., $\cos\varphi_+ = \frac{1}{2}\beta$ and $\sin\varphi_+ = \frac{1}{2}\sqrt{4-\beta^2}$. Substituting (6.2) and (6.3) into (6.1) gives

$$K_n(\lambda N) \sim \sqrt{\frac{2}{\pi}}\, n^{-1/2} p^n \sigma^{-\lambda N} e^{n(\beta^2/4+1/2+\gamma)} (4-\beta^2)^{-1/4} \times [a_0 \cos\psi_n + b_0 \cos(\psi_n + \varphi_+)]. \tag{6.5}$$

As in [8], we now write $\Phi(z_\pm) = P \pm iQ$. Since $z_\pm = \cos\varphi_+ \pm i \sin\varphi_+$, it can be easily verified that

$$a_0 = \frac{P\sin\varphi_+ - Q\cos\varphi_+}{\sin\varphi_+}, \qquad b_0 = \frac{Q}{\sin\varphi_+}, \tag{6.6}$$

and hence

$$a_0 \cos\psi_n + b_0 \cos(\psi_n + \varphi_+) = \operatorname{Re}[\Phi(z_+)\, e^{i\psi_n}].$$

The last equation can be written as

$$a_0 \cos\psi_n + b_0 \cos(\psi_n + \varphi_+) = |\Phi(z_+)| \cos(\arg\Phi(z_+) + \psi_n). \tag{6.7}$$

Since $z_\pm \leftrightarrow w_\pm$, from (4.27) we have

$$\Phi(z_\pm) = z_\pm \frac{w'(z_\pm)}{w(z_\pm)} = \frac{z_\pm}{w_\pm} w'(z_\pm). \tag{6.8}$$

To obtain the value of $w'(z_\pm)$, we let $z \to z_\pm$ and $w \to w_\pm$ in (5.3). The result is

$$w'(z_\pm) = \frac{\nu}{1-\nu}(1-w_\pm)(\sigma + w_\pm)\frac{w_\pm}{z_\pm}\frac{z_+ - z_-}{w_+ - w_-}\frac{1}{w'(z_\pm)},$$

which in turn gives

$$w'(z_{\pm}) = \left[\frac{v}{1-v} (1-w_{\pm})(\sigma + w_{\pm}) \frac{w_{\pm}}{z_{\pm}} \frac{z_{+} - z_{-}}{w_{+} - w_{-}} \right]^{1/2}. \tag{6.9}$$

Coupling (6.8) and (6.9) yields

$$\Phi(z_{\pm}) = \left[\frac{v}{1-v} (1-w_{\pm})(\sigma + w_{\pm}) \cdot \frac{z_{\pm}}{w_{\pm}} \frac{z_{+} - z_{-}}{w_{+} - w_{-}} \right]^{1/2}. \tag{6.10}$$

For notational convenience, we put

$$\left[\frac{1}{w_{\pm}} (1-w_{\pm})(\sigma + w_{\pm}) \right]^{1/2} \equiv \rho_0 e^{\pm i\alpha_0} \tag{6.11}$$

so that

$$\Phi(z_{+}) = \rho_0 \left[\frac{v}{1-v} \cdot \frac{\sin \varphi_{+}}{v_{+}} \right]^{1/2} e^{i(\alpha_0 + \varphi_{+}/2)}, \tag{6.12}$$

where $v_{+} = \operatorname{Im} w_{+}$, and

$$a_0 \cos \psi_n + b_0 \cos(\psi_n + \varphi_{+}) = \rho_0 \left[\frac{v}{1-v} \cdot \frac{\sin \varphi_{+}}{v_{+}} \right]^{1/2} \cos \left(\psi_n + \alpha_0 + \frac{\varphi_{+}}{2} \right). \tag{6.13}$$

Since $\sin \varphi_{+} = \frac{1}{2} \sqrt{4 - \beta^2}$, inserting (6.13) in (6.5), we obtain

$$K_n(\lambda N) \sim \rho_0 \left(\frac{v}{1-v} \cdot \frac{1}{\pi v_{+}} \right)^{1/2} n^{-1/2} p^n \sigma^{-\lambda N} e^{n(\beta^2/4 + 1/2 + \gamma)} \times \sin \left[n \left(\frac{1}{4} \beta \sqrt{4 - \beta^2} - \varphi_{+} \right) + \alpha_0 + \frac{3}{4} \pi \right]. \tag{6.14}$$

Note that from (4.3) and (4.4), we also have

$$\Psi(z_{+}, \beta) = (\tfrac{1}{4} \beta^2 + \tfrac{1}{2}) + i(\tfrac{1}{4} \beta \sqrt{4 - \beta^2} - \varphi_{+}).$$

Hence, by (4.6),

$$\frac{1}{4} \beta^2 + \frac{1}{2} + \gamma = \operatorname{Re}[\Psi(z_{+}, \beta) + \gamma] = \frac{1}{v} \operatorname{Re} F(w_{+}, \lambda)$$

and

$$\frac{1}{4}\beta\sqrt{4-\beta^2}-\varphi_+=\mathrm{Im}[\Psi(z_+,\beta)]=\frac{1}{v}\,\mathrm{Im}\,F(w_+,\lambda).$$

(Here we have used the fact that γ is real; this can be proved as in [8].) The asymptotic approximation (6.14) can now be written as

$$K_n(\lambda N)\sim\rho_0\left(\frac{v}{1-v}\cdot\frac{1}{\pi v_+}\right)^{1/2}n^{-1/2}p^n\sigma^{-\lambda N}$$
$$\times e^{N\,\mathrm{Re}\,F(w_+,\lambda)}\sin\left[N\,\mathrm{Im}\,F(w_+,\lambda)+\alpha_0+\frac{3}{4}\pi\right].$$

It can be readily checked that this formula agrees with Eq. (4.9) in Ismail and Simeonov [5].

The arguments for cases (ii) and (iii) are very similar to that of case (i). Hence they will not be presented here, and we shall simply state the final results.

Case (ii). For $\lambda_+<\lambda\leqslant 1-\varepsilon$, $\delta\leqslant v<q$ and $\lambda\nrightarrow\lambda_+$, we have from (6.1), (6.10), (4.3), and (4.6)

$$K_n(\lambda N)\sim\left[\frac{1}{2\pi n}\cdot\frac{v}{1-v}\cdot\frac{(1-w_-)(\sigma+w_-)}{w_-(w_+-w_-)}\right]^{1/2}p^n\sigma^{-\lambda N}e^{NF(w_-,\lambda)}. \quad (6.15)$$

For $\varepsilon\leqslant\lambda<\lambda_-$, $\delta\leqslant v<p$ and $\lambda\nrightarrow\lambda_-$, the result is

$$K_n(\lambda N)\sim\left[\frac{1}{2\pi n}\,\frac{v}{1-v}\,\frac{(1-w_+)(\sigma+w_+)}{(-w_+)(w_+-w_-)}\right]^{1/2}p^n\sigma^{-\lambda N}e^{NF(w_+,\lambda)}. \quad (6.16)$$

These two formulas correspond to Eqs. (4.36) and (4.37) in [5].

Case (iii). For $\lambda_+<\lambda\leqslant 1-\varepsilon$ and $q<v\leqslant 1-\delta$, we have

$$K_n(\lambda N)\sim\left[\frac{2}{n\pi}\cdot\frac{v}{1-v}\cdot\frac{(w_--1)(w_-+\sigma)}{w_-(w_+-w_-)}\right]^{1/2}p^n\sigma^{-\lambda N}$$
$$\times e^{N\,\mathrm{Re}\,F(w_-^+,\lambda)}[-\sin(N\,\mathrm{Im}\,F(w_-^+,\lambda))]. \quad (6.17)$$

Since $w_->1$ in this case (see Fig. 2b), we get from (3.1) and (3.2)

$$\mathrm{Re}\,F(w_-^+,\lambda)=\log\frac{(w_--1)^{1-\lambda}(w_-+\sigma)^{\lambda}}{w_-^{v}}$$

and $\operatorname{Im} F(w_-^+, \lambda) = -\pi(1-\lambda)$. Thus, (6.17) can also be expressed in the form

$$K_n(\lambda N) \sim (-1)^{N+1} \left[\frac{2}{N\pi(1-v)} \frac{(w_- - 1)(w_- + \sigma)}{w_-(w_+ - w_-)} \right]^{1/2} \times \left[\frac{(w_- - 1)^{1-\lambda} (w_- + \sigma)^{\lambda}}{\sigma^{\lambda}(w_-/p)^{v}} \right]^{N} \sin(N\pi\lambda). \tag{6.18}$$

For $\varepsilon \leqslant \lambda < \lambda_-$ and $p < v \leqslant 1 - \delta$, the result is

$$K_n(\lambda N) \sim \left[\frac{2}{n\pi} \frac{v}{1-v} \frac{(1-w_+)(-\sigma - w_+)}{(-w_+)(w_+ - w_-)} \right]^{1/2} p^n \sigma^{-\lambda N} \times e^{N \operatorname{Re} F(w_+^+, \lambda)} [-\sin(N \operatorname{Im} F(w_+^+, \lambda))], \tag{6.19}$$

or equivalently

$$K_n(\lambda N) \sim (-1)^{n+1} \left[\frac{2}{N\pi(1-v)} \cdot \frac{(1-w_+)(-\sigma - w_+)}{(-w_+)(w_+ - w_-)} \right]^{1/2} \times \left[\frac{(1-w_+)^{1-\lambda} (-\sigma - w_+)^{\lambda}}{\sigma^{\lambda}(-w_+/p)^{v}} \right]^{N} \sin(N\pi\lambda). \tag{6.20}$$

The approximations in (6.18) and (6.20) again agree with those obtained by Ismail and Simeonov; see their Eqs. (4.34) and (4.32).

Case (iv). When $\delta \leqslant v < q$, $\lambda \to \lambda_+$ corresponds to $\beta \to 2$; cf. Figs. 3 and 4. Put

$$a_+ \equiv a_+(n) \equiv n^{2/3}(\lambda - \lambda_+) \tag{6.21}$$

so that

$$\lambda = \lambda_+ + a_+ n^{-2/3}. \tag{6.22}$$

LEMMA. *When $a_+(n) = O(1)$, $n^{2/3}(\beta - 2)$ is bounded as $n \to \infty$.*

Proof. A straightforward calculation from (2.12) and (2.14) gives

$$w_\pm(\lambda) = r_0 \pm b_1 n^{-1/3} + b_2 n^{-2/3} \pm b_3 n^{-1} + O(n^{-5/3}), \tag{6.23}$$

where r_0 is given in (2.18), $b_1 = (pv)^{-1/2} r_0^{3/2} a_+^{1/2}$, $b_2 = \frac{1}{2}(pv)^{-1} r_0^2 a_+$ and $b_3 = \frac{1}{8}(pv)^{-3/2} r_0^{5/2} a_+^{3/2}$. Inserting (6.23) in (2.6), we get

$$F(w_\pm(\lambda), \lambda) = F(r_0, \lambda) \pm \tfrac{2}{3} C_+ a_+^{3/2} n^{-1} + O(n^{-5/3}), \tag{6.24}$$

where

$$C_{+}=\frac{1}{(1-r_0)(\sigma+r_0)}(1+\sigma)^{3/2}(\sigma v)^{-1/2}\,r_0^{3/2}. \tag{6.25}$$

On the other hand, from (4.3) and (4.4) we have

$$\Psi(z_+,\beta)-\Psi(z_-,\beta)=\tfrac{4}{3}(\beta-2)^{3/2}\,[1+O(\beta-2)]; \tag{6.26}$$

see [8, (6.15)]. Therefore, by (4.10),

$$(\beta-2)^{3/2}\,[1+O(\beta-2)]=C_+v^{-1}a_+^{3/2}n^{-1}+O(n^{-5/3}),$$

which in turn gives

$$\beta-2=\left(\frac{C_+}{v}\right)^{2/3}a_+n^{-2/3}[1+O(n^{-2/3})], \tag{6.27}$$

thus proving the lemma. ∎

Since $n^{2/3}(\beta-2)$ is bounded when $n^{2/3}(\lambda-\lambda_+)=O(1)$, we can use the following results in [8, (6.17) and (6.18)],

$$V_n(\beta\sqrt{n})=\sqrt{2\pi}\,e^{n(\beta^2/4-1/2)}n^{n/2+1/6}[\mathrm{Ai}(n^{2/3}(\beta-2))+O(n^{-1/3})] \tag{6.28}$$

and

$$\frac{1}{\sqrt{n}}V_n'(\beta\sqrt{n})=\sqrt{2\pi}\,e^{n(\beta^2/4-1/2)}n^{n/2+1/6}\left[\mathrm{Ai}(n^{2/3}(\beta-2))\frac{\beta}{2}+O(n^{-1/3})\right], \tag{6.29}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function. Substituting these into (6.1), we obtain

$$K_n(\lambda N)=p^n\sigma^{-\lambda N}n^{-1/3}e^{n(\beta^2/4+1/2+\gamma)}\Phi(z_-)$$
$$\times[\mathrm{Ai}(n^{2/3}(\beta-2))+O(n^{-1/3})]. \tag{6.30}$$

As in (6.26), it can be shown that

$$\Psi(z_-,\beta)=\tfrac{3}{2}+(\beta-2)-\tfrac{2}{3}(\beta-2)^{3/2}+O((\beta-2)^2).$$

Hence,

$$n(\tfrac{1}{4}\beta^2+\tfrac{1}{2}+\gamma)=n[\Psi(z_-,\beta)+\gamma+\tfrac{2}{3}(\beta-2)^{3/2}+O((\beta-2)^2)].$$

From (4.9) and (6.24), it follows that

$$n(\tfrac{1}{4}\beta^2+\tfrac{1}{2}+\gamma)=NF(r_0,\lambda)+O(n^{-1/3}). \tag{6.31}$$

In a similar manner, we have

$$z_{\pm}=1\pm(\beta-2)^{1/2}+O(n^{-2/3});$$

see [8, (6.21)]. By (6.27) and (6.23), we obtain

$$z_{+}-z_{-}=2C_{+}^{1/3}v^{-1/3}a_{+}^{1/2}n^{-1/3}+O(n^{-2/3}) \tag{6.32}$$

and

$$\begin{aligned} w_{+}-w_{-}&=2b_1 n^{-1/3}+O(n^{-1}) \\ &=2\left(\frac{1+\sigma}{\sigma v}\right)^{1/2} r_0^{3/2}a_{+}^{1/2}n^{-1/3}+O(n^{-1}). \end{aligned} \tag{6.33}$$

Inserting (6.32) and (6.33) in (6.10), and simplifying the resulting expression, we get

$$\Phi(z_{-})\sim\left[\frac{(1-r_0)(\sigma+r_0)}{\sigma}\right]^{1/3}. \tag{6.34}$$

A combination of (6.30), (6.31), and (6.34) yields

$$K_n(\lambda N)\sim\left[\frac{(1-r_0)(\sigma+r_0)}{\sigma n}\right]^{1/3} p^n\sigma^{-\lambda N}e^{NF(r_0,\,\lambda)}\,\mathrm{Ai}(a_{+}v^{-2/3}C_{+}^{2/3}) \tag{6.35}$$

or equivalently

$$K_n(\lambda N)\sim\left[\frac{(1-r_0)(\sigma+r_0)}{\sigma n}\right]^{1/3}\left[\frac{(1-r_0)^{1-\lambda}\,(\sigma+r_0)^{\lambda}}{\sigma^{\lambda}(r_0/p)^{\nu}}\right]^{N}\mathrm{Ai}(a_{+}v^{-2/3}C_{+}^{2/3}). \tag{6.36}$$

The corresponding result for the case $\lambda\to\lambda_{-}$, $n^{2/3}(\lambda-\lambda_{-})=O(1)$ and $\delta\leqslant v<p$ is given by

$$K_n(\lambda N)\sim\left[\frac{(1+r_0)(\sigma-r_0)}{\sigma n}\right]^{1/3} p^n\sigma^{-\lambda N}e^{NF(-r_0,\,\lambda)}\,\mathrm{Ai}(-a_{-}v^{-2/3}C_{-}^{2/3}) \tag{6.37}$$

or equivalently

$$\begin{aligned} K_n(\lambda N)=(-1)^n&\left[\frac{(1+r_0)(\sigma-r_0)}{\sigma n}\right]^{1/3} \\ &\left[\frac{(1+r_0)^{1-\lambda}\,(\sigma-r_0)^{\lambda}}{\sigma^{\lambda}(r_0/p)^{\nu}}\right]^{N}\mathrm{Ai}(-a_{-}v^{-2/3}C_{-}^{2/3}), \end{aligned} \tag{6.38}$$

where

$$a_{-} \equiv n^{2/3}(\lambda - \lambda_{-}) \tag{6.39}$$

and

$$C_{-} = \frac{1}{(1+r_0)(\sigma - r_0)} (1+\sigma)^{3/2} (\sigma v)^{-1/2} r_0^{3/2}. \tag{6.40}$$

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